Posterior Cramér-Rao Lower Bounds for Extended Target Tracking with Random Matrices

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***Abstract*—This paper presents posterior Cramér-Rao lower bounds (PCRLB) for extended target tracking (ETT) when the extent states of the targets are represented with random matrices. PCRLB recursions are derived for kinematic and extent states taking complicated expectations involving Wishart and inverse Wishart distributions. For some analytically intractable expectations, Monte Carlo integration is used. The bounds for the semi-major and minor axes of the extent ellipsoid are obtained as well as those for the extent matrix elements. The resulting bounds are compared on simulations with the performance of a state-of-the-art ETT algorithm employing random matrices for extent estimation.**

I. Introduction

In conventional target tracking using radars, tracking filters are developed with point target assumption. This means that, in these algorithms, it is implicitly assumed that the radar resolution is low compared to target extent. On the other hand, the improvements in radar technology has already rendered this assumption invalid. Nowadays, increased radar resolution usually leads to multiple measurements for a single target in a single scan. The number and the spread of the measurements generated depend on the target's size. This new information obtained via multiple measurements enables the estimation of target's extent along with the kinematic state. This research area of target tracking is known as extended target tracking (ETT).

In the literature, there exist different ways of modeling the target extent. In one of the earliest examples, Salmond [1] et al. takes care of the target extent by trying to track the measurement sources on the target assuming that they are stationary over the extent. This might not be possible in many practical cases since the locations of the measurement sources on the target may change rather too quickly for obtaining good quality tracks. To overcome this problem, more recent approaches model the measurement sources as random samples from a distribution whose parameters are of interest. This approach was first proposed in [2], [3] where the extent is modeled with a spatial distribution. In [4] Koch proposed a Bayesian approach modeling the measurement sources on the target to be normally distributed where the covariance of the Gaussian measurement likelihood is related to the target extent which is represented by an inverse Wishart distributed positive definite matrix. This approach, called as *random matrices*, has spurred a lot of interest in tracking community recently [5]–[10]. Baum and Hanebeck combined the idea of spatial distributions with set theory in [11] to propose the so called *random hypersurface modeling* which allows tracking of (almost) arbitrary shaped objects. A comparison of random hypersurface and random matrix methodologies is given in [12].

The tracking methods/filters used in the scope of ETT to obtain kinematic and extent estimates vary in a large range. Conventional methods such as (extended/unscented) Kalman filters [2], [13]; particle filters [2], [14], [15] as well as track before detect methods [16] are used for single ETT. A survey of Bayesian sequential Monte Carlo methods used in ETT is given in [17]. Multiple target trackers utilized for ETT are (Cardinalized) probability hypothesis density filters [18]–[24], generalized labelled multi-Bernoulli filter [25] and probabilistic multiple hypothesis trackers [26], [27].

In contrast to the variety of ETT methods, the literature on performance limits of ETT algorithms is restricted to only few works. Ristic and Salmond consider an ellipsoidal target having a constant angle between its major axis and target-observer line of sight in [13] where posterior Cramér-Rao lower bounds (PCRLBs) are obtained for making a performance comparison of extended/unscented Kalman filter. Xu and Li propose a hybrid CRLB in [28] for extended targets with rectangular shape whose state vectors contain both deterministic parameters and random kinematic variables. In [29] it is shown that PCRLB for ETT is always lower than PCRLB for conventional point target tracking on a generic ETT model and the results are illustrated on a target with elliptical extent whose parameters are to be estimated. In [30], the study [29] is expanded by considering spatial distributions for modeling the target extent. Meng et al. investigate the effect of false alarms and missed detections on the PCRLB in [31].

None of the aforementioned studies on the performance limits of the ETT algorithms uses the random matrix framework. In this study, we determine the performance limits for ETT where the target extent is modeled using random matrices. For this purpose, we first obtain the Fisher information available in the measurements about the kinematic and extent states. Then, we derive the PCRLB recursions for the kinematic states, entries of the extent matrix and its semi-major and semi-minor axes. The resulting bounds are compared to RMS errors of the random matrix based ETT filter of Feldmann et al. [5] on simulations.

The organization of the paper is given as follows. The problem formulation and the description of the ETT filter of Feldmann et al. [5] are presented in Section II. In Section III, we calculate the Fisher information matrix (FIM) for ETT. The FIM derived in Section III is used in obtaining PCRLB recursions which are the main results of the current work and given in Section IV. PCRLB values for the semi-major and minor axis of the target extent are derived in Section V. In Section VI simulation results comparing the ETT filter performance to the proposed PCRLBs are illustrated. The paper ends with the conclusions in Section VII.

II. Problem Formulation

We consider the following extended target model proposed in [5] by Feldmann et al. The state of the extended target is composed of the kinematic state $x_k \in \mathbb{R}^n$ (at time k) and the extent state $X_k \in \mathbb{R}^{d\times d}$ (at time k) where X_k is a positive definite matrix representing the ellipsoidal target extent. It is assumed that the set of measurements $Y_k \triangleq \{y_k^1, \cdots, y_k^{m_k}\}$ is collected at time k about the target and $y_k^i \in \mathbb{R}^d$, $i = 1, \ldots, m_k$, are distributed as

$$y_k^i \sim \mathcal{N}(y_k^i; Hx_k, sX_k + R) \tag{1}$$

where

- $H \in \mathbb{R}^{d\times n}$ is the measurement matrix;
- $s \in \mathbb{R}_{>0}$ is an extent scaling factor;
- R is the measurement covariance;
- the notation $\mathcal{N}(x; \bar{x}, P)$ denotes the Gaussian distribution for the random variable x with mean $\bar{x}$ and covariance P.

The number of measurements $m_k \sim p_m(\cdot)$ is assumed to be random and independent of both the kinematic and extent states. The measurements y_k^i, $i = 1, \ldots, m_k$ are assumed to be independent and identically distributed, hence, we have the following likelihood function.

$$p(Y_k|x_k, X_k) = p_m(m_k) \prod_{i=1}^{m_k} \mathcal{N}(y_k^i; Hx_k, sX_k + R). \tag{2}$$

The following state space model is assumed about the kinematic state x_k.

$$x_{k+1} = f(x_k) + w_{k+1} \tag{3}$$

where $f(\cdot)$ is a differentiable, in general, nonlinear function and $w_k \sim \mathcal{N}(w_k; 0, Q)$ is the white process noise. The initial kinematic state x_0 is random and distributed as $x_0 \sim p_{x_0}(\cdot)$. With the model (3), we have the following kinematic state transition density.

$$x_{k+1} \sim p(x_{k+1}|x_k) \triangleq \mathcal{N}(x_{k+1}; f(x_k), Q). \tag{4}$$

The extent state is assumed to be independent of the kinematic state with the transition model given as

$$X_{k+1} \sim p(X_{k+1}|X_k) \triangleq \mathcal{W}\left(X_{k+1}; n_{k+1}, \frac{X_k}{n_{k+1}}\right) \tag{5}$$

where the notation $\mathcal{W}(X, w, W)$ denotes the Wishart distribution for the random matrix X with degrees of freedom w and scale matrix W. The initial extent state X_0 is random and distributed as $X_0 \sim p_{X_0}(\cdot)$.

The aim of the current study is to obtain the posterior CRLB for the kinematic and extent states described with the model given above.

A. ETT Filter of Feldmann et al.

In [5], the posterior probability density function (pdf) of x_k and X_k is approximated as

$$p(x_k, X_k|Y_{0:k}) \approx \mathcal{N}(x_k; x_{k|k}, P_{k|k})\mathcal{IW}(X_k; v_{k|k}, V_{k|k}) \tag{6}$$

where the notation $\mathcal{IW}(X; v, V)$ denotes the inverse Wishart distribution for the random matrix X with degrees of freedom v and scale matrix V. The following Bayesian updates are proposed for the parameters $x_{k|k}$, $P_{k|k}$, $v_{k|k}$ and $V_{k|k}$ in [5].

Prediction Update:

$$x_{k+1|k} = Fx_{k|k} \tag{7a}$$
$$P_{k+1|k} = FP_{k|k}F^T + Q \tag{7b}$$
$$X_{k+1|k} = X_{k|k} \tag{7c}$$
$$\alpha_{k+1|k} = 2 + \exp(-T/\tau)(\alpha_{k|k} - 2) \tag{7d}$$

where F is the state transition matrix of the state dynamics (which is assumed to be linear); $\alpha_{k|k} \triangleq v_{k|k} - d - 1$; the parameter τ is a forgetting time-constant for the extent; T is the sampling time.

Measurement Update:

$$x_{k+1|k+1} = x_{k+1|k} + K_{k+1|k}(\bar{y}_{k+1} - Hx_{k+1|k}) \tag{8a}$$
$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k}S_{k+1|k}K_{k+1|k}^T \tag{8b}$$
$$X_{k+1|k+1} = \frac{1}{\alpha_{k+1|k+1}}\left(\alpha_{k+1|k}X_{k+1|k} + \hat{N}_{k+1|k} + \hat{Y}_{k+1|k}\right)$$
$$\alpha_{k+1|k+1} = \alpha_{k+1|k} + n_{k+1} \tag{8c}$$

where

$$\bar{y}_{k+1} = \frac{1}{m_{k+1}} \sum_{i=1}^{m_{k+1}} y_{k+1}^i, \tag{9a}$$
$$S_{k+1|k} = HP_{k+1|k}H^T + \frac{Y_{k+1|k}}{m_{k+1}}, \tag{9b}$$
$$K_{k+1|k} = P_{k+1|k}H^T S_{k+1|k}^{-1}, \tag{9c}$$
$$Y_{k+1|k} = sX_{k+1|k} + R, \tag{9d}$$
$$\bar{Y}_{k+1} = \sum_{i=1}^{m_{k+1}} \left(y_{k+1}^i - \bar{y}_{k+1}\right)\left(y_{k+1}^i - \bar{y}_{k+1}\right)^T, \tag{9e}$$
$$N_{k+1|k} = \left(\bar{y}_{k+1} - Hx_{k+1|k}\right)\left(\bar{y}_{k+1} - Hx_{k+1|k}\right)^T, \tag{9f}$$
$$\hat{N}_{k+1|k} = X_{k+1|k}^{1/2} S_{k+1|k}^{-1/2} N_{k+1|k} \left(S_{k+1|k}^{-1/2}\right)^T \left(X_{k+1|k}^{1/2}\right)^T,$$
$$\hat{Y}_{k+1|k} = X_{k+1|k}^{1/2} Y_{k+1|k}^{-1/2} \bar{Y}_{k+1} \left(Y_{k+1|k}^{-1/2}\right)^T \left(X_{k+1|k}^{1/2}\right)^T. \tag{9g}$$

We compare the PCRLB results obtained in this work with the performance of the ETT filter described above in the simulations.

III. FIM for Extended Target Tracking

In this section, as an intermediate result, we calculate the Fisher information contained in the measurement set Y_k about the kinematic and extent states, x_k and X_k. For this purpose, we constrain the model proposed in Section II to 2D. In other words, we assume that the measurements y_k^i are $\mathsf{x}-\mathsf{y}$ measurements, i.e., $d = 2$.

We first define the augmented state ξ_k as

$$\xi_k \triangleq \begin{bmatrix} \xi_k^x \\ \xi_k^X \end{bmatrix} \tag{10}$$

where

$$\xi_k^x \triangleq x_k, \tag{11}$$

$$\xi_k^X \triangleq \begin{bmatrix} [X_k]_{11} & [X_k]_{12} & [X_k]_{22} \end{bmatrix}^T. \tag{12}$$

Here, the notation $[\cdot]_{ij}$ denotes the element of the argument matrix corresponding to the ith row and the jth column. In the definition of ξ_k^X in (12), the entry $[X_k]_{21}$ is omitted, because the extent matrix X_k is symmetric, i.e., $[X_k]_{12} = [X_k]_{21}$. The likelihood function $p(Y_k|\xi_k)$ can be written by making the following substitutions into (2).

$$x_k \leftarrow \xi_k^x, \tag{13}$$

$$X_k \leftarrow \begin{bmatrix} [\xi_k^X]_1 & [\xi_k^X]_2 \\ [\xi_k^X]_2 & [\xi_k^X]_3 \end{bmatrix}, \tag{14}$$

where the notation $[\cdot]_i$ denotes the ith element of the argument vector. The Fisher information matrix (FIM), denoted by $\mathcal{I}_k$, is defined as

$$\mathcal{I}_k \triangleq E\left[-\Delta_{\xi_k}^{\xi_k} \log p(Y_k|\xi_k)\right]. \tag{15}$$

An equivalent expression, which involves only first-order derivatives, is given as follows.

$$\mathcal{I}_k \triangleq E\left[\nabla_{\xi_k} \log p(Y_k|\xi_k) \nabla_{\xi_k}^T \log p(Y_k|\xi_k)\right] \tag{16}$$

$$= E\left[S_{\xi_k} S_{\xi_k}^T\right]. \tag{17}$$

where the vector S_{ξ_k} is called as the score function and defined as follows.

$$S_{\xi_k} \triangleq \nabla_{\xi_k} \log p(Y_k|\xi_k). \tag{18}$$

A. Calculation of the Score Function

In this section, we drop the subscripts k for brevity. The score function S_ξ can be written as

$$S_\xi = \begin{bmatrix} S_{\xi^x} \\ S_{\xi^X} \end{bmatrix} \tag{19}$$

where

$$S_{\xi^x} \triangleq \nabla_{\xi^x} \log p(Y|\xi), \tag{20}$$

$$S_{\xi^X} \triangleq \nabla_{\xi^X} \log p(Y|\xi). \tag{21}$$

Since we have $\xi^x = x$, we can write the kinematic score function S_{ξ^x} as

$$S_{\xi^x} = \nabla_x \log p(Y_k|x, X) \triangleq S_x. \tag{22}$$

Noting the definition of ξ^X in (12), the extent score function S_{ξ^X} is given as follows.

$$S_{\xi^X} = \begin{bmatrix} [S_X]_{11} \\ [S_X]_{12} + [S_X]_{21} \\ [S_X]_{22} \end{bmatrix}, \tag{23}$$

where we have

$$S_X \triangleq \nabla_X \log p(Y|x, X). \tag{24}$$

Note that the score S_{ξ^X} is a vector of size 3 which contains the derivatives of the log-likelihood $\log p(Y|x, X)$ with respect to the 3 unique elements of the matrix X, i.e., with respect to $[X]_{11}$, $[X]_{12}$ and $[X]_{22}$ under the symmetry constraint for X, i.e., $[X]_{21} = [X]_{12}$. On the other hand, the score S_X is a matrix of size 2×2 which is composed of the partial derivatives of the log-likelihood $\log p(Y|x, X)$ with respect to the elements of the matrix X (without the symmetry constraint).

The score functions S_x and S_X can be found as follows (See [32, Section 3.2.1] for a detailed derivation).

$$S_x = H^T (sX + R)^{-1} \sum_{p=1}^{m} (y^p - Hx), \tag{25}$$

$$S_X = \frac{s}{2}(sX + R)^{-1} \left(\sum_{p=1}^{m} (y^p - Hx)(y^p - Hx)^T \right) \times (sX + R)^{-1} - 0.5sm(sX + R)^{-1}. \tag{26}$$

B. Fisher Information Matrix

The FIM $\mathcal{I}_k$ given in (17) can be expressed in terms of the kinematic and extent score functions S_x and S_{ξ^X} as follows.

$$\mathcal{I}_k = \begin{bmatrix} E\left[S_x S_x^T\right] & E\left[S_x S_{\xi^X}^T\right] \\ E\left[S_{\xi^X} S_x^T\right] & E\left[S_{\xi^X} S_{\xi^X}^T\right] \end{bmatrix}. \tag{27}$$

The blocks of the FIM can be computed as given below (See [32, Section 3.2.2] for a detailed derivation).

$$E[S_x S_x^T] = \bar{m} H^T (sX + R)^{-1} H, \tag{28a}$$

$$E\left[S_x S_{\xi^X}^T\right] = 0, \tag{28b}$$

$$E\left[S_{\xi^X} S_x^T\right] = 0 \tag{28c}$$

where $\bar{m}$ is the expected value of the number of measurements. The lower-right block of the FIM is given as

$$E\left[S_{\xi^X} S_{\xi^X}^T\right] = \begin{bmatrix} d_{11,11} & d_{11,12} + d_{11,21} & d_{11,22} \\ \begin{matrix} d_{12,11} \\ +d_{21,11} \end{matrix} & \begin{matrix} d_{12,12} + d_{12,21} \\ +d_{21,12} + d_{21,21} \end{matrix} & \begin{matrix} d_{12,22} \\ +d_{21,22} \end{matrix} \\ d_{22,11} & d_{22,12} + d_{22,21} & d_{22,22} \end{bmatrix} \tag{28d}$$

where the terms $d_{ij,kl}$, which are defined as

$$d_{ij,kl} \triangleq E\left[[S_X]_{ij} [S_X]_{kl}\right], \tag{29}$$

can be calculated as follows.

$$d_{ij,kl} = \frac{s^2}{4}\bar{m}\left(\left[(sX+R)^{-1}\right]_{ik}\left[(sX+R)^{-1}\right]_{jl} + \left[(sX+R)^{-1}\right]_{il}\left[(sX+R)^{-1}\right]_{kj}\right). \tag{30}$$

It is important to emphasize that the resulting FIM, i.e., $\mathcal{I}_k$ in (27), is a block diagonal matrix.

IV. PCRLB Recursion

In this section we give PCRLB recursions for the kinematic and extent states using the approach proposed by Tichavsky et al. in [33]. Since

- the initial kinematic and extent states are assumed independent in Section II;
- the kinematic and extent state transitions are assumed independent in Section II;
- the FIM calculated in III is block-diagonal,

it can be seen that the recursions of the PCRLB for kinematic and extent states are independent. These recursions are described separately in the following.

A. PCRLB Recursion for the Kinematic State

The PCRLB for the kinematic state is shown as $\tilde{\mathcal{J}}_k^x$ and is given as

$$\tilde{\mathcal{J}}_k^x = \left(\tilde{\mathcal{I}}_k^x\right)^{-1} \tag{31}$$

where the Bayesian FIM $\tilde{\mathcal{I}}_k^x$ for the kinematic state can be calculated using the following recursion.

$$\tilde{\mathcal{I}}_0^x = E\left[\nabla_{x_0}\log p_{x_0}(x_0)\nabla_{x_0}^T\log p_{x_0}(x_0)\right], \tag{32a}$$

$$\tilde{\mathcal{I}}_{k+1}^x = D_{k+1}^{x,22} + E\left[\mathcal{I}_{k+1}^{11}(x_{k+1},X_{k+1})\right] - D_{k+1}^{x,21}\left(\tilde{\mathcal{I}}_k^x + D_{k+1}^{x,11}\right)^{-1}D_{k+1}^{x,12}, \tag{32b}$$

where $\mathcal{I}_{k+1}^{11} \triangleq E[S_x S_x^T]$ is the FIM calculated for the kinematic state given in (28a) and

$$D_{k+1}^{x,11} \triangleq E\left[\nabla_{x_k}\log p(x_{k+1}|x_k)\nabla_{x_k}^T\log p(x_{k+1}|x_k)\right],$$
$$D_{k+1}^{x,12} \triangleq E\left[\nabla_{x_k}\log p(x_{k+1}|x_k)\nabla_{x_{k+1}}^T\log p(x_{k+1}|x_k)\right],$$
$$D_{k+1}^{x,21} \triangleq E\left[\nabla_{x_{k+1}}\log p(x_{k+1}|x_k)\nabla_{x_k}^T\log p(x_{k+1}|x_k)\right],$$
$$D_{k+1}^{x,22} \triangleq E\left[\nabla_{x_{k+1}}\log p(x_{k+1}|x_k)\nabla_{x_{k+1}}^T\log p(x_{k+1}|x_k)\right].$$

Using the transition pdf given in (4), the information submatrices defined above can be calculated as follows.

$$D_{k+1}^{x,11} = E\left[F^T(x_k)Q^{-1}F(x_k)\right], \tag{34a}$$
$$D_{k+1}^{x,12} = -E\left[F^T(x_k)\right]Q^{-1}, \tag{34b}$$
$$D_{k+1}^{x,21} = -Q^{-1}E\left[F(x_k)\right], \tag{34c}$$
$$D_{k+1}^{x,22} = Q^{-1}, \tag{34d}$$

where $F(\cdot)$ represents the Jacobian of $f(\cdot)$. Substituting these results into (32b) yields the following recursion.

$$\tilde{\mathcal{I}}_{k+1} = Q^{-1} - Q^{-1}E\left[F(x_k)\right] \times\left(\tilde{\mathcal{I}}_k + E\left[F^T(x_k)Q^{-1}F(x_k)\right]\right)^{-1}E^T\left[F(x_k)\right]Q^{-1} + E\left[\mathcal{I}_{k+1}^{11}(x_{k+1},X_{k+1})\right], \tag{35}$$

$$= Q^{-1} - Q^{-1}E\left[F(x_k)\right] \times\left(\tilde{\mathcal{I}}_k + E\left[F^T(x_k)Q^{-1}F(x_k)\right]\right)^{-1}E^T\left[F(x_k)\right]Q^{-1} + E\left[\bar{m}_{k+1}H^T(sX_{k+1}+R)^{-1}H\right] \tag{36}$$

Note that the expectations in (36) cannot be taken analytically for many practical systems and hence they require the use of Monte Carlo integration methods.

If the initial pdf is taken as $p_{x_0}(x_0) = \mathcal{N}(x_0;\bar{x},P_0)$; then, the initial Bayesian FIM turns out to be

$$\tilde{\mathcal{I}}_0^x = P_0^{-1}. \tag{37}$$

Thus, the initial PCRLB is given by

$$\tilde{\mathcal{J}}_0^x = P_0. \tag{38}$$

B. PCRLB Recursion for the Extent State

The PCRLB for the extent state is shown as $\tilde{\mathcal{J}}_k^X$ and is given as

$$\tilde{\mathcal{J}}_k^X = \left(\tilde{\mathcal{I}}_k^X\right)^{-1} \tag{39}$$

where the Bayesian FIM $\tilde{\mathcal{I}}_k^X$ for the extent state can be calculated using the following recursion.

$$\tilde{\mathcal{I}}_0^X = E\left[\nabla_{\xi_0^X}\log p_{X_0}(X_0)\nabla_{\xi_0^X}^T\log p_{X_0}(X_0)\right], \tag{40a}$$

$$\tilde{\mathcal{I}}_{k+1}^X = D_{k+1}^{X,22} + E\left[\mathcal{I}_{k+1}^{22}(x_{k+1},X_{k+1})\right] - D_{k+1}^{X,21}\left(\tilde{\mathcal{I}}_k^X + D_{k+1}^{X,11}\right)^{-1}D_{k+1}^{X,12}, \tag{40b}$$

where $\mathcal{I}_{k+1}^{22} \triangleq E[S_{\xi^X}S_{\xi^X}^T]$ is the FIM for the extent state given in (28d) and calculated as

$$\mathcal{I}_{k+1}^{22} = \begin{bmatrix} d_{11,11} & d_{11,12}+d_{11,21} & d_{11,22} \\ d_{12,11} + d_{21,11} & d_{12,12}+d_{0,12,21} + d_{21,12}+d_{0,21,21} & d_{12,22} + d_{21,22} \\ d_{22,11} & d_{22,12}+d_{22,21} & d_{22,22} \end{bmatrix}$$

where

$$d_{ij,lm} \triangleq \frac{s^2}{4}\bar{m}_{k+1}\left(\left[(sX_{k+1}+R)^{-1}\right]_{il}\left[(sX_{k+1}+R)^{-1}\right]_{jm} + \left[(sX_{k+1}+R)^{-1}\right]_{im}\left[(sX_{k+1}+R)^{-1}\right]_{lj}\right).$$

The other information submatrices in 40b are defined as

$$D_{k+1}^{X,11} \triangleq E\left[\nabla_{\xi_k^X}\log p(X_{k+1}|X_k)\nabla_{\xi_k^X}^T\log p(X_{k+1}|X_k)\right],$$
$$D_{k+1}^{X,12} \triangleq E\left[\nabla_{\xi_k^X}\log p(X_{k+1}|X_k)\nabla_{\xi_{k+1}^X}^T\log p(X_{k+1}|X_k)\right],$$
$$D_{k+1}^{X,21} \triangleq E\left[\nabla_{\xi_{k+1}^X}\log p(X_{k+1}|X_k)\nabla_{\xi_k^X}^T\log p(X_{k+1}|X_k)\right],$$
$$D_{k+1}^{X,22} \triangleq E\left[\nabla_{\xi_{k+1}^X}\log p(X_{k+1}|X_k)\nabla_{\xi_{k+1}^X}^T\log p(X_{k+1}|X_k)\right].$$

Using a similar method applied for the FIM of the extent state given in (28d), the information submatrices defined above can be written as follows.

$$D_{k+1}^{X,tu} = \begin{bmatrix} d_{11,11}^{tu} & d_{11,12}^{tu} + d_{11,21}^{tu} & d_{11,22}^{tu} \\ d_{12,11}^{tu} & d_{12,12}^{tu} + d_{12,21}^{tu} & d_{12,22}^{tu} \\ +d_{21,11}^{tu} & +d_{21,12}^{tu} + d_{21,21}^{tu} & +d_{21,22}^{tu} \\ d_{22,11}^{tu} & d_{22,12}^{tu} + d_{22,21}^{tu} & d_{22,22}^{tu} \end{bmatrix} \quad (41)$$

where $t, u \in \{1, 2\}$ and

$$d_{ij,lm}^{11} \triangleq E\begin{bmatrix} [\nabla_{X_k} \log p(X_{k+1}|X_k)]_{ij} \\ \times [\nabla_{X_k} \log p(X_{k+1}|X_k)]_{lm} \end{bmatrix}, \quad (42a)$$

$$d_{ij,lm}^{12} \triangleq E\begin{bmatrix} [\nabla_{X_k} \log p(X_{k+1}|X_k)]_{ij} \\ \times [\nabla_{X_{k+1}} \log p(X_{k+1}|X_k)]_{lm} \end{bmatrix}, \quad (42b)$$

$$d_{ij,lm}^{21} \triangleq E\begin{bmatrix} [\nabla_{X_{k+1}} \log p(X_{k+1}|X_k)]_{ij} \\ \times [\nabla_{X_k} \log p(X_{k+1}|X_k)]_{lm} \end{bmatrix}, \quad (42c)$$

$$d_{ij,lm}^{22} \triangleq E\begin{bmatrix} [\nabla_{X_{k+1}} \log p(X_{k+1}|X_k)]_{ij} \\ \times [\nabla_{X_{k+1}} \log p(X_{k+1}|X_k)]_{lm} \end{bmatrix}. \quad (42d)$$

The terms defined in (42) can be calculated as given below (See [32, Section 4.2.2] for a detailed derivation).

$$\begin{aligned} d_{ij,lm}^{11} =& \frac{n_{k+1}}{4}\Big(E\left[[X_k^{-1}]_{il}[X_k^{-1}]_{jm}\right] \\ &+E\left[[X_k^{-1}]_{im}[X_k^{-1}]_{lj}\right]\Big) \quad (43a) \\ d_{ij,lm}^{21} =& -\frac{n_{k+1}}{4}\Big(E\left[[X_k^{-1}]_{il}[X_k^{-1}]_{jm}\right] \\ &+E\left[[X_k^{-1}]_{jl}[X_k^{-1}]_{im}\right]\Big) \quad (43b) \\ d_{ij,lm}^{12} =& d_{lm,ij}^{21} \quad (43c) \\ d_{ij,lm}^{22} =& \frac{n_{k+1}^2}{4}\left(c_1 (n_{k+1}-d-1)^2 - 1\right) \\ &\times E\left[[X_k^{-1}]_{ij}[X_k^{-1}]_{lm}\right] \\ &+ c_2 \frac{n_{k+1}^2}{4}(n_{k+1}-d-1)^2 E\left[[X_k^{-1}]_{il}[X_k^{-1}]_{jm}\right] \\ &+ c_2 \frac{n_{k+1}^2}{4}(n_{k+1}-d-1)^2 E\left[[X_k^{-1}]_{lj}[X_k^{-1}]_{im}\right] \end{aligned} \quad (43d)$$

where constants c_1 and c_2 used above are given as

$$c_1 = c_2(n_{k+1} - d - 2), \quad (44a)$$

$$c_2 = [(n_{k+1} - d)(n_{k+1} - d - 1)(n_{k+1} - d - 3)]^{-1}. \quad (44b)$$

Note that the expectations in (43) cannot be taken analytically and hence they require the use of Monte Carlo integration methods.

If the pdf of the initial extent state is assumed to be as follows,

$$p_{X_0}(X_0) = \mathcal{W}\left(X_0; \bar{n}, \frac{\bar{X}}{\bar{n}}\right), \quad (45)$$

where the constant $\bar{n}$ and the matrix $\bar{X}$ are known, the following initial Bayesian FIM is obtained.

$$\tilde{\mathcal{I}}_0^X = \begin{bmatrix} d_{0,11,11} & d_{0,11,12} + d_{0,11,21} & d_{0,11,22} \\ d_{0,12,11} & d_{0,12,12} + d_{0,12,21} & d_{0,12,22} \\ +d_{0,21,11} & +d_{0,21,12} + d_{0,21,21} & +d_{0,21,22} \\ d_{0,22,11} & d_{0,22,12} + d_{0,22,21} & d_{0,22,22} \end{bmatrix} \quad (46)$$

where the terms $d_{0,ij,kl}$ are given as

$$\begin{aligned} d_{0,ij,kl} =& \frac{\bar{n}^2}{4}\left(c_1(\bar{n}-d-1)^2 - 1\right)[\bar{X}^{-1}]_{ij}[\bar{X}^{-1}]_{kl} \\ &+ c_2 \frac{\bar{n}^2(\bar{n}-d-1)^2}{4} \\ &\times \left([\bar{X}^{-1}]_{ik}[\bar{X}^{-1}]_{jl} + [\bar{X}^{-1}]_{kj}[\bar{X}^{-1}]_{il}\right) \end{aligned} \quad (47)$$

where constants c_1 and c_2 used above are given as

$$c_1 = c_2(\bar{n} - d - 2), \quad (48a)$$

$$c_2 = [(\bar{n} - d)(\bar{n} - d - 1)(\bar{n} - d - 3)]^{-1}. \quad (48b)$$

V. PCRLB for Semi-Major and Semi-Minor Axes

In this section, based on the PCRLB matrices $\tilde{\mathcal{J}}_k^X$ calculated for the target extent matrix X in the previous section, we obtain the PCRLB for the (lengths of) semi-major and semi-minor axes of the ellipsoidal target extent. The general PCRLB theory says that when a parameter θ with posterior CLRB $\tilde{\mathcal{J}}^\theta$ is transformed with a differentiable, in general, nonlinear function $g(\cdot)$, the PCRLB $\tilde{\mathcal{J}}^{\bar{\theta}}$ for the transformed parameter $\bar{\theta} \triangleq g(\theta)$ is given as follows [34, Section 1.2.8.2].

$$\tilde{\mathcal{J}}^{\bar{\theta}} = E\left[\nabla_\theta^T g(\theta)\right]\tilde{\mathcal{J}}^\theta E\left[\nabla_\theta g(\theta)\right]. \quad (49)$$

The semi-major and semi-minor axes of the target extent, denoted as a_{major} and a_{minor} respectively, can be written in terms of the elements of ξ^X in (12) (under the symmetry constraint for X, i.e., $[X]_{21} = [X]_{12}$) as follows.

$$\begin{aligned} a_{\text{major}}^2(X) =& 0.5\Big([X]_{11} + [X]_{22} \\ &+ \sqrt{([X]_{11} + [X]_{22})^2 - 4\left([X]_{11}[X]_{22} - [X]_{12}^2\right)}\Big), \end{aligned} \quad (50a)$$

$$\begin{aligned} a_{\text{minor}}^2(X) =& 0.5\Big([X]_{11} + [X]_{22} \\ &- \sqrt{([X]_{11} + [X]_{22})^2 - 4\left([X]_{11}[X]_{22} - [X]_{12}^2\right)}\Big). \end{aligned} \quad (50b)$$

The gradients of $a_{\text{major}}(\cdot)$ and $a_{\text{minor}}(\cdot)$ given above with respect to the elements of ξ^X in (12) can be computed as below.

$$\nabla_{\xi^X} a_{\text{major}}(X) = \frac{1}{2a_{\text{major}}(X)} \begin{bmatrix} \frac{1}{2} + \frac{X_{11}-X_{22}}{2\sqrt{(\operatorname{tr} X)^2-4\det X}} \\ 2\frac{X_{12}}{\sqrt{(\operatorname{tr} X)^2-4\det X}}, \\ \frac{1}{2} + \frac{X_{22}-X_{11}}{2\sqrt{(\operatorname{tr} X)^2-4\det X}} \end{bmatrix}, \tag{51a}$$

$$\nabla_{\xi^X} a_{\text{minor}}(X) = \frac{1}{2a_{\text{minor}}(X)} \begin{bmatrix} \frac{1}{2} - \frac{X_{11}-X_{22}}{2\sqrt{(\operatorname{tr} X)^2-4\det X}} \\ -2\frac{X_{12}}{\sqrt{(\operatorname{tr} X)^2-4\det X}} \\ \frac{1}{2} - \frac{X_{22}-X_{11}}{2\sqrt{(\operatorname{tr} X)^2-4\det X}} \end{bmatrix}. \tag{51b}$$

Using (49), the PCRLBs for semi-major and semi-minor axes are obtained as below.

$$\tilde{\mathcal{J}}_k^{a_{\text{major}}} = E\left[\nabla^T_{\xi_k^X} a_{\text{major}}(X_k)\right] \tilde{\mathcal{J}}_k^X E\left[\nabla_{\xi_k^X} a_{\text{major}}(X_k)\right], \tag{52a}$$

$$\tilde{\mathcal{J}}_k^{a_{\text{minor}}} = E\left[\nabla^T_{\xi_k^X} a_{\text{minor}}(X_k)\right] \tilde{\mathcal{J}}_k^X E\left[\nabla_{\xi_k^X} a_{\text{minor}}(X_k)\right]. \tag{52b}$$

Note that the expectations in (52) cannot be taken analytically and hence they require the use of Monte Carlo integration methods.

VI. Simulations

A. Simulation Parameters

We generate random target kinematic and extent state trajectories whose initial states are drawn from the following pdfs,

$$x_0 \sim \mathcal{N}(\bar{x}_0, P_0), \tag{53a}$$

$$X_0 \sim \mathcal{W}_2\left(n_0, \frac{\overline{X}_0}{n_0}\right), \tag{53b}$$

where

$$\bar{x}_0 = [0\,\text{m}, 0\,\text{m}, 500\,\text{m/s}, 500\,\text{m/s}]^T, \tag{54a}$$

$$P_0 = \operatorname{diag}\left[75\,\text{m}^2, 75\,\text{m}^2, 15\,\text{m}^2/\text{s}^2, 15\,\text{m}^2/\text{s}^2\right], \tag{54b}$$

$$n_0 = 20000. \tag{54c}$$

The matrix $\overline{X}_0$ is selected as

$$\overline{X}_0 = E\Lambda E^T \tag{55}$$

where

$$\Lambda = \begin{bmatrix} a^2_{\text{major}} & 0 \\ 0 & a^2_{\text{minor}} \end{bmatrix}, \tag{56a}$$

$$E = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{56b}$$

$$\theta = 45^\circ, \tag{56c}$$

$$a_{\text{major}} = 300, \tag{56d}$$

$$a_{\text{minor}} = 100. \tag{56e}$$

The true target kinematic and extent states evolve randomly with the following transition pdfs.

$$p(x_{k+1}|x_k) = \mathcal{N}(x_{k+1}; Fx_k, GQG^T), \tag{57a}$$

$$p(X_{k+1}|X_k) = \mathcal{W}\left(X_{k+1}; n_{k+1}, \frac{X_k}{n_{k+1}}\right), \tag{57b}$$

where the matrices F and G are given as

$$F \triangleq \begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G \triangleq \begin{bmatrix} T^2/2 & 0 \\ 0 & T^2/2 \\ T & 0 \\ 0 & T \end{bmatrix} \tag{58}$$

with $T = 1\,\text{s}$. The process noise covariance is set as $Q = \sigma_v^2 I_2$ where I_ℓ denotes and identity matrix of size $\ell \times \ell$ and $\sigma_v^2 = 1\,\text{m}^2/\text{s}^4$. The parameter n_k is selected as $n_k = 20000$ for all k, which corresponds to a case where the target extent is almost constant. The total duration of the scenario is 100 s.

The measurements are generated randomly in accordance with the likelihood function (1) where

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad R = \sigma_v^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{59}$$

with $\sigma_v^2 = 1000\,\text{m}^2$ and $s = 1$. The simulations are carried out using deterministic number of measurements, i.e., the distribution of the number of measurements is given as $p_m(m') = \delta_m(m')$ where the notation $\delta_m(\cdot)$ denotes the Dirac delta function placed at $m' = m$. The number of measurements is set to values $m = 5$, $m = 20$ and $m = 80$ in different experiments.

The PCRLB matrices are computed using Monte Carlo integration for the analytically intractable expectations mentioned in Sections III and IV. The number of MC runs to calculate the expectations is set to be $N_{\text{mc}} = 100000$.

The ETT filter proposed by Feldmann et al. and described in Section II-A is implemented to obtain kinematic and extent state estimates from the noisy measurements generated as described above. The algorithm is executed on $N_{\text{mc}} = 100000$ random sets of measurements. The algorithm uses the true target model parameters except that n_0 is set to 10 for initialization purposes. The initial degree of freedom variable is set to $\alpha_{0|-1} = 2.1$ and the extent forgetting time is selected as $\tau = 10\,\text{s}$. RMS errors for the kinematic states, extent states, and semi-major/minor axes of the extent ellipsoid are obtained.

B. Results

Figure 1 illustrates the RMS position and velocity errors with the square roots of the corresponding PCRLBs for different number of measurements. As expected PCRLBs decrease as the number of measurements increase. This is a manifestation of the fact that the recursion of FIM is an affine function of the expected number of measurements. We observe that the kinematic RMS errors follow the PCRLBs almost perfectly. This shows that the ETT algorithm of Feldmann et al. is a nearly optimal estimator for the kinematic states. Figure 2 shows the RMS extent matrix errors along with the square roots of the corresponding PCRLBs. It is seen that the RMS errors do not follow PCRLBs as closely as in the case of the kinematic states, especially for low number of measurements. Thus, if PCRLB is assumed to be tight, i.e., reachable by a practical estimator, better estimators can be found for extent estimation. The results for the semi-major and semi-minor axes shown in Figure 3 are similar.

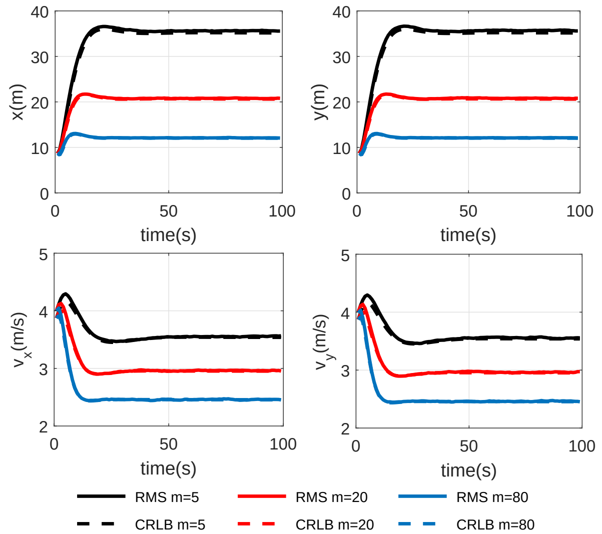

Fig. 1. RMS errors and square-root PCRLB values for kinematic states with $m = 5$, $m = 20$ and $m = 80$.

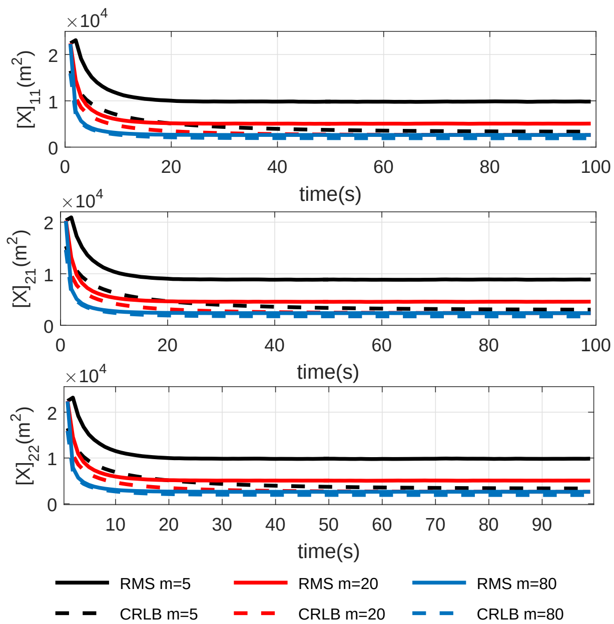

Fig. 2. RMS errors and square-root PCRLB values for extent matrix elements with $m = 5$, $m = 20$ and $m = 80$.

In the simulation scenario above, the target extent matrix has the transition density $p(X_{k+1}|X_k)$ (57b) with the parameter $n_k = 20000$. This selection, as aforementioned, corresponds to an almost constant extent along time, i.e., mathematically we have $p(X_{k+1}|X_k) \approx \delta(X_{k+1} - X_k)$ where $\delta(\cdot)$ stands for the Dirac delta function. On the other hand, while obtaining the results presented above, the extent forgetting time-constant τ, which is a parameter defined as "a time constant related to the agility with which the object may change its extension over time" in [5], was selected as $\tau = 10\,\text{s}$ in the ETT filter.

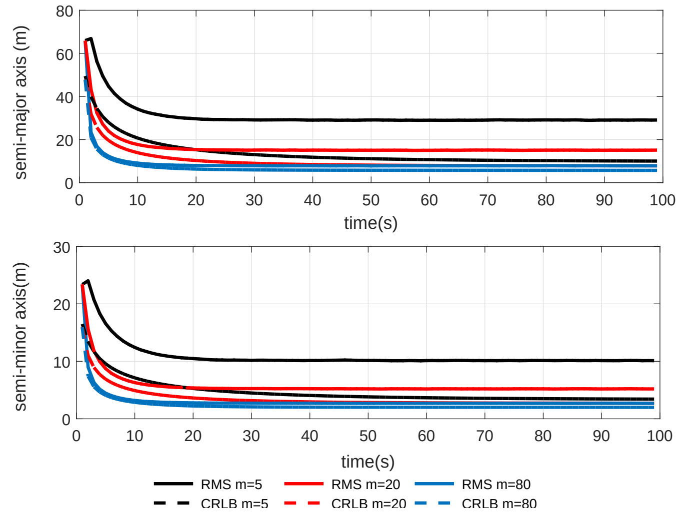

Fig. 3. RMS errors and square-root PCRLB values for semi-major and minor axes with $m = 5$, $m = 20$ and $m = 80$.

Hence, there is a mismatch between the true extent model and the extent model used in the ETT filter. In order to examine the effect of τ on the performance of the ETT filter, we repeated the MC runs using the values $\tau = 5\,\text{s}$, $\tau = 10\,\text{s}$, $\tau = 20\,\text{s}$ and $\tau = 100\,\text{s}$ when the number of measurements is set to $m = 5$. It has been seen that the value of τ does not have much effect on the kinematic estimation performances. Figures 4 and 5 show the results for the extent matrix elements and semi-major/minor axes respectively. It is apparent that the increase in τ yields significant reductions in the estimation errors. This is a consequence of the fact that the true extent model and the extent model used in the filter becomes more and more similar as τ increases. For $\tau = 100\,\text{s}$, it is observed that the RMS errors become very close to the PCRLB values in the steady-state. Hence it can be said that the ETT filter of Feldmann et al. is almost optimal in the steady-state under model match conditions though it still seems that there is room for improvement in the transient behavior especially under model mismatch.

VII. Conclusion

This paper has presented PCRLBs for ETT to be used when the target extent is modeled using random matrices. The comparisons between the calculated PCRLBs and the ETT filter of Feldmann et al. show that this filter

- is almost optimal for the kinematic states;
- is almost optimal for the extent states under model match conditions in the steady-state.

The discrepancies between the PCRLBs and the RMS errors of the filter for the extent states during the transients and under model-mismatch suggest that there might be better ETT estimators for these conditions.

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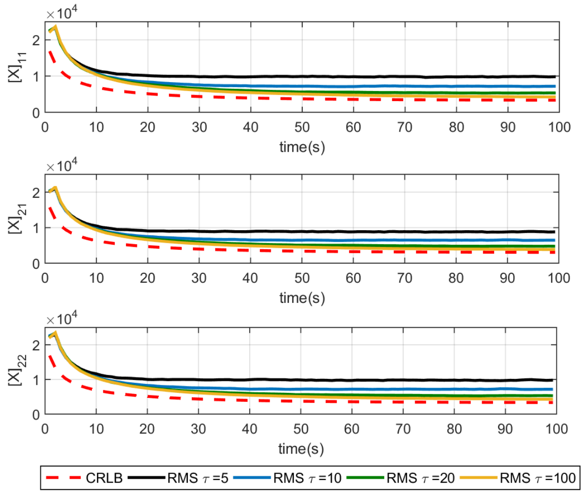

Fig. 4. RMS errors and square-root PCRLBs for extent matrix elements with $\tau = 5\,\mathrm{s}$, $\tau = 10\,\mathrm{s}$, $\tau = 20\,\mathrm{s}$ and $\tau = 100\,\mathrm{s}$ when $m = 5$.

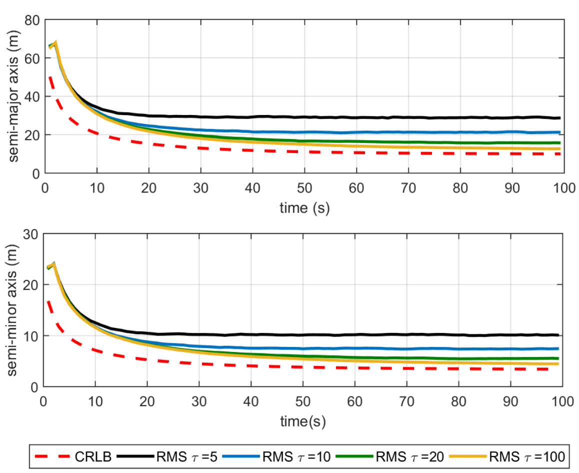

Fig. 5. RMS errors and square-root PCRLBs for semi-major and minor axes with $\tau = 5\,\mathrm{s}$, $\tau = 10\,\mathrm{s}$, $\tau = 20\,\mathrm{s}$ and $\tau = 100\,\mathrm{s}$ when $m = 5$.

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